GENERALIZED EXTREMAL ELEMENTS FOR LINEAR FUNCTIONALS IN BANACH SPACES

D.A. NOMIROVSKII

Let E be a Banach space, M be a bounded closed subset of E, and φ be a bounded continuous functional defined on the set E. If E is an infinite-dimensional Banach space, the set M may cease to be a compact set, and the continuous functional $\varphi(x)$, $x \in M$, may have no maximal and minimal points in the set M. Thus, we have a problem how to find such continuous extension of the functional φ from the set M on some new set $\bar{M}$ ($M \subset \bar{M}$, $\bar{M} \not\subset E$) that there exist "generalized extremal elements" $x_{max}, x_{min} \in \bar{M}$ for the extended functional $\bar{\varphi}$. This problem can be solved in the following way. Consider the simplest case when M is the unit ball $S_1(E)$ of the space E. Let the Banach space E be densely and compactly embedded in a Banach space F. Then the conjugate Banach space F^* is embedded in the Banach space E^*. Let $\overline{S_1(E)}$ be the closure of the unit ball $S_1(E) \subset E$ in the space F. Since $E \subset F$ is a compact embedding, the set $\overline{S_1(E)}$ is a compact set in the space F. Thus, there exists such an element $\bar{x}^* \in \overline{S_1(E)} \subset F$ that

$$\|f\|_{E^*} = \sup_{x \in S_1(E)} |f(x)| = \sup_{x \in \overline{S_1(E)}} |f(x)| = |f(\bar{x}^*)|$$

as the restriction of the functional $f \in F^*$ from the set F on the set $\overline{S_1(E)}$ is a continuous functional. So the elements $\bar{x}^* \in F^*$ and $(-\bar{x}^*) \in F^*$ are generalized extremal elements for the following problems

$$f(x) \to \max\,(\min), \quad x \in S_1(E).$$

Observe that if the Banach space E is not a reflexive space, then by James' theorem there exists such a functional $f \in E^*$ that $f \in E^*$ has no maximal and minimal points in the unit ball $S_1(E)$. In the case when this functional f is a member of F^* there exists a generalized maximal (and minimal) element $\bar{x}^*$ in the set $\overline{S_1(E)} \subset F^*$. Moreover, we can specify such an intermediate Banach space H ($E \subset H \subset F$), namely

$$H = \{\, x \in F \mid \exists R > 0, x_n \in E : \|x_n\|_E \leq R, \|x_n - x\|_F \to 0 \,\},$$
$$\|x\|_H = \inf \{\, R \in \mathbb{R} \mid \exists x_n \in E : \|x_n\|_E \leq R, \|x_n - x\|_F \to 0 \,\},$$

that $\bar{x}^* \in H$.

To realize this plan we need the following theorem

Theorem 1. *For every separable Banach space E there exists such a separable Banach space F that the space E is embedded in the space F densely and compactly.*

References

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Department of Computational Mathematics, National Taras Shevchenko University of Kyiv, Kyiv, Glushkova 2, Ukraine

E-mail address: kashpir@mail.ru